

The Jacobi-metric for timelike geodesics in static spacetimes

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Abstract

It is shown that the free motion of massive particles moving in static spacetimes are given by the geodesics of an energy-dependent Riemannian metric on the spatial sections analogous to Jacobi's metric in classical dynamics. In the massless limit Jacobi's metric coincides with the energy independent Fermat or optical metric. For stationary metrics, it is known that the motion of massless particles is given by the geodesics of an energy independent Finslerian metric of Randers type. The motion of massive particles is governed by neither a Riemannian nor a Finslerian metric. The properties of the Jacobi metric for massive particles moving outside the horizon of a Schwarzschild black hole are described. By contrast with the massless case, the Gaussian curvature of the equatorial sections is not always negative.

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1 Introduction

An elegant device for implementing the Principle of Least Action of Maupertuis was introduced by Jacobi. One varies the action of a mechanical system

$$\int_{\gamma} p_i dx^i = \int_{\gamma} p_i \dot{x}^i dt \quad (1.1)$$

along an unparameterized path γ in an n -dimensional configuration space Q with coordinates $x^i, i = 1, 2, \dots, n$ and canonical momenta p_i subject to the constraint that along the curve γ the energy E is conserved. An equivalent formulation is to lift the curve γ to the cotangent space T^*Q and restrict variations to a level set of the Hamiltonian $H(x, p) = E$.

In the simplest case the kinetic energy $T = \frac{1}{2}m_{ij}(x)\dot{x}^i\dot{x}^j$, where the space dependent mass matrix $m_{ij}(x)dx^i dx^j$ endows the configuration space Q with a Riemannian metric and the Lagrangian giving the equations of motion is

$$L = \frac{1}{2}m_{ij}(x)\dot{x}^i\dot{x}^j - V(x). \quad (1.2)$$

Jacobi showed that the unparamaterised curves extremizing the constrained action are geodesics of the rescaled Jacobi metric

$$j_{ij}(E, x)dx^i dx^j = 2(E - V)m_{ij}dx^i dx^j. \quad (1.3)$$

One recovers the parameterization of the motion as a function of physical time t by noting that length s with respect to the Jacobi metric is related to the original time parameter t by

$$dt = \frac{ds}{2(E - V)}. \quad (1.4)$$

This procedure opened up the way open to investigations of the motion of the original mechanical system using the methods developed by differential geometers to investigate geodesic motion. Of particular interest is the influence of the curvature of the Jacobi metric [1, 2]. An important application to gravity was the work by Ong [1] who studied the curvature of the the Jacobi metric for the Newtonian N -body problem (see also [3]). One has $Q = \mathbb{R}^{3n}$ with the flat metric

$$m_{ij}dx^i dx^j = \sum_a m_a d\mathbf{x}_a^2, \quad (1.5)$$

and the potential energy is

$$V = - \sum_{1 \leq a < b \leq N} \frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|}. \quad (1.6)$$

Since in this case, the Jacobi metric is conformally flat the evalution of the curvature is straight forward. If $N = 2$, the problem reduces to the Kepler's problem of the relative motion and the relevant Jacobi metric is up to an unimportant over all constant factor ¹.

$$\left(\frac{E}{m} + \frac{M}{r}\right) (dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi)^2) \quad (1.7)$$

By symmetry, one may restrict attention to the equatorial plane $\theta = \frac{\pi}{2}$ which is a totally geodesic submanifold. One then has a 2 -dimensional axially symmetric metric

$$\left(\frac{E}{m} + \frac{M}{r}\right)(dr^2 + r^2d\phi^2). \quad (1.8)$$

Ong [1] showed that the sign of the Gaussian curvature of the metric has the opposite sign to that of the energy E . If $E > 0$, which of course corresponds to unbound hyperbolic or parabolic orbits, he showed that the Jacobi metric (1.8) is well defined and complete for $0 \leq r < \infty$ and if $E < 0$, which corresponds to bound elliptical orbits, it is well defined for $0 < r < \frac{2Mm}{E}$.

Ong also gave an isometric embedding of the Jacobi manifold into three dimensional Euclidean space \mathbb{E}^3 with Cartesian coordinates x, y, z as a surface of revolution $z = f(\sqrt{x^2 + y^2})$. If $E = 0$, then $f'' = 0$ and the surface is the cone $z = \sqrt{3}\sqrt{x^2 + y^2}$ which has deficit angle π , or equivalently, semi-angle 30° . In the other two cases the surface approaches the cone near the origin. If $E > 0$, then $f'' < 0$; the surface has negative Gauss curvature and remains outside the cone . If $E < 0$, then $f'' > 0$ the surface remains inside the cone and asymptotes the cylinder $\sqrt{x^2 + y^2} = \frac{2Mm}{-E}$. Ong also studied the three body problem using these techniques.

It is obviously of interest, if only to extend one's intuition by means of an easily visualised model, to see whether these ideas can be applied to General Relativity. At a formal level, one takes the configuration space $\{Q, m_{ij}\}$ to be Wheeler's superspace equipped with its DeWitt metric (cf. [4]). In the vacuum case, the potential V is

$$\int_{\Sigma} R\sqrt{g} d^3x \quad (1.9)$$

¹Note that E in this Newtonian case does not contain a contribution from the rest mass of the particle.

and the Hamiltonian constraint implies that the energy vanishes $E = 0$. One then obtains a picture of spacetime as a sheaf of geodesics in superspace. Having obtained the geodesic between two points in superspace one obtains the time duration between them using (1.4), thus solving the much discussed “problem of time”.

Less ambitiously one may confine attention to a mini-superspace truncation, and this has been done in attempts to investigate inflation [5] and the chaotic behaviour of Bianchi IX Mixmaster models (see e.g.[6]).

The focus of the present paper is different. It is the motion a test particle of rest mass m following a timelike geodesic in a stationary background. A limiting case would be a zero rest-mass particle. This latter case is well known in the static case to reduce to geodesic motion with respect to the optical or Fermat metric $f_{ij}(x) = \frac{1}{-g_{tt}}g_{ij}$. This was studied in [7, 8] and a number of limitations on possible motions using the Gauss-Bonnet theorem. In particular in [8] the Gaussian curvature of the Schwarzschild optical metric restricted to the equatorial plane was shown to be everywhere negative and to approach a constant value near the horizon $r = 2M$. In [9] this behaviour was found to be universal for the near horizons of non-extreme static black holes. One purpose of the present paper is to extend this work to the case of massive particles. The extension of Fermat’s principle to cover stationary spacetimes entails replacing the Riemmanian metric by a Finsler metric of Randers type (see e.g. [10, 11, 12]).

2 The Jacobi metric for static spacetimes

If

$$ds = -V^2 dt^2 + g_{ij} dx^i dx^j, \quad (2.1)$$

the action for a massive particle is

$$S = -m \int L dt = -m \int dt \sqrt{V^2 - g_{ij} \dot{x}^i \dot{x}^j} \quad (2.2)$$

where $\dot{x}^i = \frac{dx^i}{dt}$. The canonical momentum is

$$p_i = \frac{m \dot{x}^i}{\sqrt{V^2 - g_{ij} \dot{x}^i \dot{x}^j}}. \quad (2.3)$$

whence the Hamiltonian is

$$H = \frac{mV^2}{\sqrt{V^2 - g_{ij}\dot{x}^i\dot{x}^j}} \quad (2.4)$$

$$= \sqrt{m^2V^2 + V^2g^{ij}p_ip_j}. \quad (2.5)$$

Setting

$$p_i = \partial_i S, \quad (2.6)$$

the Hamilton-Jacobi equation becomes

$$\sqrt{m^2V^2 + V^2g^{ij}\partial_i S \partial_j S} = E \quad (2.7)$$

or

$$f^{ij}\partial_i S \partial_j S = E^2 - m^2V^2, \quad (2.8)$$

where $f^{ij}f_{jk} = \delta_k^i$ and

$$f_{ij} = V^{-2}g_{ij} \quad (2.9)$$

is the optical or Fermat metric. Thus

$$\frac{1}{E^2 - m^2V^2}f^{ij}\partial_i S \partial_j S = 1, \quad (2.10)$$

which is the Hamilton-Jacobi equation for geodesics of the Jacobi-metric j_{ij} given by

$$j_{ij}dx^i dx^j = (E^2 - m^2V^2)V^{-2}g_{ij}dx^i dx^j. \quad (2.11)$$

Note that the massless case, $m = 0$, the Jacobi metric coincides with the Fermat metric up to a factor of E^2 and as a consequence the geodesics, considered as unparameterized curves, do not depend upon the energy E . However in the massive case, $m \neq 0$ the geodesics *do* depend upon E .

In general, if the spacetime is asymptotically flat and the sources obey the energy conditions, then $0 \leq V \leq 1$. Therefore, if $E^2 \geq m^2$, the Jacobi metric is positive definite and complete, even if horizons are present. If however $E^2 < m^2$ there are bound orbits and the Jacobi metric changes signature at large distances. Generically there will be a level set of V on which $E^2 - m^2V^2$ vanishes and hence on which the Jacobi metric vanishes. From the point of view of the Jacobi metric this level set is a point-like conical singularity. Every geodesic must have a turning point on or inside this level set.

3 The Schwarzschild Case

In the case of the Schwarzschild solution the Jacobi metric is

$$ds^2 = \left(E^2 - m^2 + \frac{2Mm^2}{r} \right) \left(\frac{dr^2}{(1 - \frac{2M}{r})^2} + \frac{r^2}{(1 - \frac{2M}{r})} (d\theta^2 + \sin^2 \theta d\phi^2) \right). \quad (3.1)$$

The first large bracket in (3.1) is the conformal factor and the second large bracket the optical metric. The later is defined for $2M < r < \infty$ and the horizon at $r = 2M$ is infinitely far way with respect to the radial optical radial distance or tortoise coordinate

$$r^* = \int_{2M}^r \frac{dx}{1 - \frac{2M}{x}} = r - 2M + \ln\left(\frac{r}{2M} - 1\right), \quad \Leftrightarrow \quad r - 2M = 2MW(e^{\frac{r^*}{2M}}), \quad (3.2)$$

where $W(x)$ is Lambert's function defined by $\ln x = W(x) + \ln W(x)$.

By spherical symmetry, in order to study geodesics, it is sufficient to consider the equatorial plane $\theta = \frac{\pi}{2}$ on which the restriction of the Jacobi metric is

$$ds^2 = \left(E^2 - m^2 + \frac{2Mm^2}{r} \right) \left(\frac{dr^2}{(1 - \frac{2M}{r})^2} + \frac{r^2}{(1 - \frac{2M}{r})} d\phi^2 \right). \quad (3.3)$$

By axi-symmetry we have a conserved quantity often called Clairaut's constant which corresponds physically to angular momentum. That is

$$l = \left(E^2 - m^2 + \frac{2Mm^2}{r} \right) \frac{r^2}{(1 - \frac{2M}{r})} \frac{d\phi}{ds} = \text{constant}. \quad (3.4)$$

Now

$$\left(E^2 - m^2 + \frac{2Mm^2}{r} \right) \left(\left(\frac{dr}{ds} \right)^2 \frac{1}{(1 - \frac{2M}{r})^2} + \frac{r^2}{(1 - \frac{2M}{r})} \left(\frac{d\phi}{ds} \right)^2 \right) = 1. \quad (3.5)$$

whence

$$\left(E^2 - m^2 + \frac{2Mm^2}{r} \right)^2 \frac{1}{(1 - \frac{2M}{r})^2} \left(\frac{dr}{ds} \right)^2 = E^2 - \left(1 - \frac{2M}{r} \right) \left(m^2 + \frac{l^2}{r^2} \right), \quad (3.6)$$

which agrees with the standard result that

$$m^2 \left(\frac{dr}{d\tau} \right)^2 = E^2 - \left(1 - \frac{2M}{r} \right) \left(m^2 + \frac{l^2}{r^2} \right), \quad (3.7)$$

where τ is proper time along the particle's worldline and

$$l = mr^2 \frac{d\phi}{d\tau} \quad (3.8)$$

as long as

$$d\tau = m \frac{1 - 2\frac{M}{r}}{E^2 - m^2 + \frac{2Mm^2}{r}} ds. \quad (3.9)$$

Thus l is indeed angular momentum. In standard treatments $u = \frac{l}{r}$ satisfies Binet's equation

$$\frac{d^2u}{d\phi^2} + u = \frac{F(u)}{h^2u^2}, \quad (3.10)$$

where, for a massive particle orbiting a Schwarzschild black hole, we have

$$\frac{F(u)}{h^2u^2} = 3Mu^2 + \frac{M}{h^2}, \quad (3.11)$$

and where $h = \frac{l}{m}$ is the conserved angular momentum per unit mass. If this were a classical central orbit problem we would say that we have a sum of an inverse fourth and inverse square law attraction. There is a first integral

$$\left(\frac{du}{d\phi}\right)^2 = -u^2 + 2Mu^3 + \frac{2Mu}{h^2} + C = 2M(u - \alpha)(u - \beta)(u - \gamma) \quad (3.12)$$

where C is a constant related to the energy per unit mass $\mathcal{E} = \frac{E}{m}$ and the angular momentum per unit mass h by

$$C = \frac{\mathcal{E}^2 - 1}{h^2} \quad (3.13)$$

with

$$\alpha + \beta + \gamma = \frac{1}{2M}, \quad \beta\gamma + \gamma\alpha + \alpha\beta = \frac{1}{h^2}, \quad \alpha\beta\gamma = -\frac{C}{2M}. \quad (3.14)$$

3.1 Some Explicit Solutions

The behaviour of the orbits depend on the two dimensionless quantities which are the specific energy $\mathcal{E} > 0$ and $\frac{M}{h}$. In general, the solutions are given by elliptic integrals. However there is a one parameter family of explicit solutions of the form

$$u = A + \frac{B}{\cosh^2(\omega\phi)} \quad (3.15)$$

$$A = \frac{1}{6M} \left(1 \pm \sqrt{1 - \frac{12M^2}{h^2}}\right) \quad (3.16)$$

$$B = \mp \frac{1}{2M} \sqrt{1 - \frac{12M^2}{h^2}} \quad (3.17)$$

$$\omega^2 = \pm \frac{1}{4} \sqrt{1 - \frac{12M^2}{h^2}} \quad (3.18)$$

$$C = A^2(4MA - 1). \quad (3.19)$$

These solutions arise because two of the the roots of the cubic eqns in (3.12) coincide.

If ω is real and $A + B \neq 0$ the solutions are symmetric about $\phi = 0$, $r = \frac{1}{A+B}$ and end spiralling around a circular geodesic at $r = \frac{1}{A}$. If $h^2 = 12M^2$ then $B = 0$ and we have the inner most stable circular orbit at $r = 6M$ for which the specific energy $\mathcal{E} = \sqrt{\frac{8}{9}}$. If $h^2 = 16M^2$ the energy per unit mass $\mathcal{E} = 1$. Since $A = -B = \frac{1}{4M}$, these orbits are in free fall from infinity, starting from rest and spiral around a circular orbit at $r = 4M$. All orbits starting from rest at infinity (i.e. having $\mathcal{E} = 1$) with $|h| < 4M$ fall through the horizon at $r = 2M$ while all such orbits with $|h| > 4M$ are scattererd back to infinty. These latter orbits are relevant for the theory of the BSW effect [13].

3.2 Bound States and Jacobi functions

For a bound orbit we have three real positive roots taken to satisfy

$$\alpha > \beta \geq u \geq \gamma > 0. \quad (3.20)$$

The first integral (3.12) leads to

$$\frac{1}{2}\varpi d\phi = \frac{\sqrt{\alpha - \gamma} du}{\sqrt{4(\alpha - u)(\beta - u)(u - \gamma)}} \quad (3.21)$$

with

$$\varpi = \sqrt{2M(\alpha - \gamma)} = \sqrt{\frac{\alpha - \gamma}{\alpha + \beta + \gamma}}. \quad (3.22)$$

Thus [16]

$$\begin{aligned} u &= \gamma + (\beta - \gamma) \operatorname{sn}^2\left(\frac{\varpi\phi}{2}\right), \\ &= \beta - (\beta - \gamma) \operatorname{cn}^2\left(\frac{\varpi\phi}{2}\right), \\ &= \alpha - (\alpha - \gamma) \operatorname{dn}^2\left(\frac{\varpi\phi}{2}\right), \end{aligned} \quad (3.23)$$

where the modulus k of the elliptic functions is given by

$$k = \sqrt{\frac{\beta - \gamma}{\alpha - \gamma}}, \quad (3.24)$$

and the quarter period K by

$$K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}. \quad (3.25)$$

Using (3.23) and properties of elliptic functions, (3.21) may expressed as

$$\frac{1}{r} = \frac{1}{r_p} \operatorname{cn}^2\left(\frac{\varpi\phi}{2}\right) + \frac{1}{r_a} \operatorname{sn}^2\left(\frac{\varpi\phi}{2}\right) = \frac{1}{L} \left(1 + e[\operatorname{cn}^2\left(\frac{\varpi\phi}{2}\right) - \operatorname{sn}^2\left(\frac{\varpi\phi}{2}\right)]\right), \quad (3.26)$$

where the constants $r_p \leq r_a$ are the radii at perihelion and aphelion respectively, since from (3.20)

$$r_p = \frac{1}{\gamma} \leq r \leq \frac{1}{\beta} = r_a \quad (3.27)$$

and

$$L = \frac{2r_p r_a}{r_p + r_a}, \quad e = \frac{r_a - r_p}{r_a + r_a}. \quad (3.28)$$

$$\text{Perihelion is at : } \phi = \frac{4K}{\varpi} n, \quad n \in \mathbb{Z} \quad (3.29)$$

$$\text{Aphelion is at : } \phi = \frac{4K}{\varpi} \left(n + \frac{1}{2}\right), \quad n \in \mathbb{Z} \quad (3.30)$$

These expressions generalise the Newton-Kepler case for which the orbits are ellipses with foci at the origin and are given by

$$\frac{1}{r} = \frac{1}{r_p} \cos^2\left(\frac{\phi}{2}\right) + \frac{1}{r_a} \sin^2\left(\frac{\phi}{2}\right) = \frac{1}{L} (1 + e \cos \phi), \quad (3.31)$$

where L is the semi-latus rectum and e is the eccentricity Note that in both cases L is the harmonic mean of the perihelion and aphelion radii.

3.3 Relation to Weierstrass Functions and Photon Orbits

The term linear in u may be eliminated by setting

$$u = v + c, \quad (3.32)$$

where

$$c^2 - \frac{c}{3M} + \frac{1}{3h^2}, \iff c = \frac{1}{6M} \left(1 \pm \sqrt{1 - \frac{4M^2}{h^2}}\right). \iff 1 - 6Mc = \mp \sqrt{1 - \frac{4M^2}{h^2}}, \quad (3.33)$$

We then find that if $\tilde{\phi} = \phi \sqrt{1 - 6Mc}$, $\tilde{M} = \frac{M}{1 - 6Mc}$, and $\tilde{C} = C - c^2(1 - 2Mc)$, then

$$\left(\frac{dv}{d\tilde{\phi}}\right)^2 = -v^2 + 2\tilde{M}v^3 + \tilde{C}, \quad (3.34)$$

which is the equation governing the orbit of a photon in a Schwarzschild solution of mass \tilde{M} . Thus the solution is [14]

$$\tilde{M}v = \frac{1}{6} + 2\wp(\tilde{\phi}), \quad (3.35)$$

where \wp is the Weierstrass's Elliptic function with parameter $g_3 = \frac{1}{216} - \left(\frac{\tilde{M}}{2}\right)^2 \tilde{C}$.

It follows that the massive particle orbits are given by

$$u = \frac{1}{6M} + \frac{1 - 6Mc}{3M} \wp(\phi \sqrt{1 - 6Mc}). \quad (3.36)$$

A particular example is provided by the cardioidal photon orbit [14] with $\tilde{C} = 0$,

$$v = \frac{1}{2\tilde{M} \cos^2(\frac{\tilde{\phi}}{2})}, \quad (3.37)$$

which gives

$$u = c + v = c + \frac{1 - 6Mc}{2M \cos^2(\frac{\sqrt{1-6Mc}\phi}{2})}. \quad (3.38)$$

If we take the case for which $1 - 6Mc > 0$ this gives

$$6Mu = 1 - \sqrt{1 - \frac{4M^2}{h^2}} + 3 \frac{\sqrt{1 - \frac{4M^2}{h^2}}}{\cos^2(\frac{(1 - \frac{4M^2}{h^2})^{\frac{1}{4}}\phi}{2})}. \quad (3.39)$$

These orbits run from the past singularity at $r = 0$ out to a maximum radius r_{\max} given by

$$r_{\max} = \frac{6M}{1 + 2\sqrt{1 - \frac{4M^2}{h^2}}}, \quad (3.40)$$

and return to the future singularity at $r = 0$. Note that $r_{\max} \geq 2M$.

For a recent treatment of photon orbits in the Schwarzschild metric using Jacobi elliptic functions, the reader is directed to [15].

3.4 Properties of the Jacobi metric

Since any spherically symmetric is conformally flat we could adopt isotropic coordinates and avail ourselves of the results given in [1]. Alternatively, the calculation of the Gaussian curvature K with respect to the Jacobi metric on the equatorial planes is straight forward using equation (8) of [9]. However unless $m = 0$, this leads to rather complicated and un-illuminating expresions. A qualititive analysis, based on

the behaviour of circular geodesics, i.e. those with $r = \text{constant}$ seems preferable. We restrict attention to an equatorial plane.

If $E^2 \geq m^2$, K tends zero at infinity and in all three cases it tends to $-\frac{1}{16M^2E^2}$ as r tends to the horizon at $r = 2M$, which is at infinite Jacobi radial distance, the curvature tends to a negative constant. If $E^2 < m^2$, the Jacobi manifold has an outer boundary at which the metric vanishes. This happens when

$$E^2 = m^2\left(1 - \frac{2M}{r}\right), \quad r = \frac{2Mm^2}{m^2 - E^2} > 2M. \quad (3.41)$$

This outer boundary should be thought of as a point since the Jacobi circumference

$$2\pi\left(E^2 - m^2\left(1 - \frac{2M}{r}\right)\right) \frac{r}{\sqrt{1 - \frac{2M}{r}}} \quad (3.42)$$

vanishes there. In the vicinity of the boundary K is positive.

Circular Jacobi geodesics are possible. These correspond to extrema of the Jacobi circumference and are located at values of r for which

$$E^2\left(1 - \frac{3M}{r}\right) - m^2\left(1 - \frac{2M}{r}\right)^2 = 0, \quad \frac{E^2}{m^2} = \frac{(r - 2M)^2}{r(r - 2M)}. \quad (3.43)$$

Circular geodesics exist with real energies of all values of $r \geq 3M$. For every value of $\frac{E^2}{m^2} > 1$ there is a unique circular geodesic with radius r between $3M$ and $4M$. For every value of $\frac{E^2}{m^2}$ between $\frac{8}{9}$ and 1 there are two circular null geodesics, the inner, which is unstable, has its radius between $4M$ and $6M$, and the outer whose radius is greater than $6M$. Three interesting cases arise

- $m^2 = 0$, $r = 3M$. These are circular null geodesics which are circular geodesics of the optical metric.
- $E^2 = m^2$, $r = 4M$, these are have the smallest radius among all circular bound geodesics.
- $E^2 = \frac{8}{9}m^2$, $r = 6M$. These are the most deeply bound circular timelike geodesics.

These results are consistent with the standard approach to circular timelike geodesics which is to require the simultaneous vanishing of the right hand side of (3.7) and its derivative. Eliminating E^2 and solving for u gives

$$ru = \frac{1}{6M} \left(1 \pm \sqrt{1 - \frac{12M^2}{h^2}}\right). \quad (3.44)$$

Eliminating E and solving for h gives

$$\frac{16M^2m^2}{h^2} = \frac{16(r - 3M)}{r^2}. \quad (3.45)$$

For the unbound circular geodesics with $E^2 > m^2$, $\frac{16M^2m^2}{h^2}$ varies from 0 to $\frac{3}{4}$ as r varies from $3M$ to $4M$. For the bound circular geodesics with $m^2 \leq E^2 \leq \frac{8}{9}m^2$ one finds that $\frac{16M^2m^2}{h^2}$ varies from $\frac{3}{4}$ to 1 as r varies from $4M$ to $6M$ where it achieves its maximum value and thereafter as r varies from $6M$ to infinity it decreases monotonically to zero.

3.5 Gauss Curvature and Isometric Embedding

We can use these results to say something about the Gauss curvature K .

We begin by recalling that if the induced metric on the surface of revolution $z = f(\rho)$, $\rho = \sqrt{x^2 + y^2}$, is

$$A^2 dr^2 + C^2(r) d\phi^2 = (1 + (f')^2) d\rho^2 + \rho^2 d\phi^2 \quad (3.46)$$

then we have

$$\rho = C(r), \quad (f')^{-1} = \left(\frac{1}{A} \frac{dC}{dr} \right)^2 - 1. \quad (3.47)$$

In our case, the embedding will extend from infinity towards the horizon at $r = 2M$ as long as the r.h.s of (3.47) remains positive. The Jacobi metric is conformal to the spatial metric of the Schwarzschild solution and its equatorial plane is well known to be isometrically embeddable as all the way down to the horizon as the Flamm paraboloid [18, 19]

$$z = \frac{(\rho - 2M)^2}{8M}. \quad (3.48)$$

However this is in general not possible for the Jacobi metric.

For example if $m = 0$ one is limited us to the region $r > \frac{9}{4}M$. In that case

$$K = -\frac{2M}{r^3} \left(1 - \frac{3M}{r} \right) \quad (3.49)$$

which is everywhere negative and near the horizon the optical metric is asymptotic to one of constant negative curvature equal to $-\frac{1}{(4M)^2}$.

This is analogous to the well known fact that the metric with $A = 1$ and $C = ae^{-\frac{r}{a}}$, $a > 0$ has constant negative curvature $-\frac{1}{a^2}$ and may be embedded into \mathbb{E}^3 as the

surface of revolution whose meridional curve is a tractrix. However this “Beltrami’s trumpet” for which

$$f' = \frac{1}{\rho} \sqrt{a^2 - \rho^2}, \quad \pm f = \text{constant} + a \cosh^{-1}\left(\frac{1}{\rho}\right) - \sqrt{a^2 - \rho^2}, \quad (3.50)$$

is incomplete and only occupies the region for which $\rho < a$, i.e. $r > 0$.

By (3.3) the Jacobi metric is conformal to the optical metric with conformal factor

$$E^2 - m^2 + \frac{2Mm^2}{r}. \quad (3.51)$$

It also tends to constant negative curvature near the horizon. Thus in general one does not expect to be able to capture the near horizon geometry of the Jacobi metric by an isometric embedding into \mathbb{E}^3 as a surface of revolution. This represents a practical obstruction to constructing black hole analogues using such materials as graphene [20].

Consider now the case $m^2 > E^2 > \frac{8}{9}m^2$. Near the outer boundary the Gauss curvature is positive and it is positive at the outer circular geodesic which is a local maximum of the Jacobi circumference. By the time we get to the inner, unstable orbit, which is a local minimum of the Jacobi circumference, the Gauss curvature is negative and it is negative near the horizon at $r = 2M$ near which the Jacobi circumference diverges. If $E^2 < \frac{8}{9}m^2$ there are no circular geodesics and the curvature is positive near the boundary and negative near the horizon. Thus the Gauss curvature of the Jacobi-metric restricted to the equatorial plane is not everywhere negative as is the case for the Fermat metric.

4 The Jacobi metric for stationary spacetimes

We cast the spacetime metric in Zermelo form [10]

$$g_{\mu\nu} dx^\mu dx^\nu = \frac{V^2}{1 - h_{ij} W^i W^j} \left[-dt^2 + h_{ij} (dx^i - W^i)(dx^j - W^j) \right], \quad (4.1)$$

where I shall call h_{ij} the Zermelo metric, W^i the wind, and

$$V^2 = -g_{\mu\nu} K^\mu K^\nu = -g_{tt}, \quad (4.2)$$

where $K^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial t}$, is the timelike Killing vector field. Note that if the wind vanishes, the Zermelo metric coincides with the optical or Fermat metric.

The Lagrangian L for a point particle of mass m undergoing geodesic motion in a spacetime with metric (4.1) is

$$L = -\frac{mV}{\sqrt{1-h_{ij}W^iW^j}}\sqrt{1-h_{ij}(\dot{x}^i-W^i)(\dot{x}^j-W^j)} \quad (4.3)$$

where $\dot{x}^i = \frac{dx^i}{dt}$. The canonical momenta p_i are therefore given by

$$p_i = \frac{mV}{\sqrt{1-h_{ij}W^iW^j}}\frac{h_{ij}(\dot{x}^j-W^j)}{\sqrt{1-h_{ij}(\dot{x}^i-W^i)(\dot{x}^j-W^j)}}. \quad (4.4)$$

The Hamiltonian, $H = p_i\dot{x}^i - L$, is given by

$$H(m, p_i, x^i) = \sqrt{h^{ij}p_ip_j + \frac{m^2}{V^2}(1-h_{ij}W^iW^j + p_iW^i)}, \quad (4.5)$$

where $h^{ik}h_{kj} = \delta_j^i$. Note that in the massless case, $m = 0$ the Hamiltonian H becomes

$$H(0, p_i, x^i) = \sqrt{h^{ij}p_ip_j} + p_iW^i \quad (4.6)$$

which coincides with equation (14) of [10] thus recovering the result that the projection of null geodesics of a stationary spacetime onto the space of orbits of the timelike Killing vector solve the Zermelo problem of minimizing the time of travel in the presence of the wind and with respect to the Zermelo metric h_{ij} .

A quick way of obtaining (4.5) is to work on the co-tangent bundle of spacetime. The so-called super-Hamiltonian \mathcal{H} is subject to the constraint

$$\mathcal{H} = g^{\mu\nu}p_\mu p_\nu = -m^2. \quad (4.7)$$

If one solves (4.7) for $p_0 = -H$ one obtains (4.5).

In general, on the level set $H = E$ of the Hamiltonian we have

$$h^{ij}p_ip_j + \frac{m^2V^2}{(1-h_{ij}W^iW^j)} = (E-p_iW^i)^2. \quad (4.8)$$

If the mass m is non-zero, it is not possible to cast this in the form of an expression which is a homogeneous degree two in momenta p_i equated to a constant. If it were so, then a Legendre transform would result in a Lagrangian which is of degree two in velocities $v^i = \dot{x}^i$ and hence we would be dealing with a Finsler structure, possibly Riemannian, as in the case of a static metric. Thus we are faced with a geometric structure more general than a Riemannian or even a Finsler metric.

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